Problema 104

(propuesto por José Luis Díaz Barrero, Barcelona, España)
Solución de Marcos Martinelli, Brasil

$$\lim_{n\to+\inf}\left(\sum_{k=1}^{n} e^{\frac{1}{n+2005k}}-n\right)=\lim_{n\to+\inf}\left[\sum_{k=1}^{n}\sum_{j=0}^{+\inf}\frac{1}{(n+2005k)^j\ j!}-n\right]=$$

$$=\lim_{n\to+\inf}\left\{\sum_{k=1}^{n}\left[1+\sum_{j=1}^{+\inf}\frac{1}{(n+2005k)^j\ j!}\right]-n\right\}=\lim_{n\to+\inf}\left[\sum_{k=1}^{n}\sum_{j=1}^{+\inf}\frac{1}{(n+2005k)^j\ j!}\right].$$

Calculemos agora o seguinte limite $\lim_{n\to+\inf}\sum_{k=1}^{n}\frac{1}{(n+2005k)^j}$, $\forall j\in\mathbb{N}$. Para isso considere a seguinte função $f:\mathbb{R}\to\mathbb{R}$ tal que $f(x)=\frac{1}{(1+2005x)^j}$. Podemos observar que $\sum_{k=1}^{n}\frac{1}{\left(1+2005\frac{k}{n}\right)^j}.\frac{1}{n}\le\int_0^1\frac{1}{(1+2005x)^j}dx$ e ainda notamos que

$\sum_{k=0}^{n}\frac{1}{\left(1+2005\frac{k}{n}\right)^j}.\frac{1}{n}-\frac{1}{\left(1+2005\frac{n}{n}\right)^j}.\frac{1}{n}\ge\int_0^1\frac{1}{(1+2005x)^j}dx$ e que, portanto:

$$\frac{1}{\left(1+2005\frac{n}{n}\right)^j}.\frac{1}{n}+\int_0^1\frac{1}{(1+2005x)^j}dx-\frac{1}{n}\le\sum_{k=1}^{n}\frac{1}{\left(1+2005\frac{k}{n}\right)^j}.\frac{1}{n}\le\int_0^1\frac{1}{(1+2005x)^j}dx\to$$

$$\frac{n}{n^j}.\left[\frac{1}{\left(1+2005\frac{n}{n}\right)^j}.\frac{1}{n}+\int_0^1\frac{1}{(1+2005x)^j}dx-\frac{1}{n}\right]\le\sum_{k=1}^{n}\frac{1}{(n+2005k)^j}\le\frac{n}{n^j}\int_0^1\frac{1}{(1+2005x)^j}dx\to$$

$\lim_{n\to+\inf}\sum_{k=1}^{n}\frac{1}{(n+2005k)^j}=\int_0^1\frac{1}{(1+2005x)^j}dx$ se j=1 e $\lim_{n\to+\inf}\sum_{k=1}^{n}\frac{1}{(n+2005k)^j}=0$ se $j\ge 2$.

Logo o limite procurado vale $\int_0^1\frac{1}{(1+2005x)}dx=\frac{\ln(2006)}{2005}$.

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