Problemas para os máis novos (23)

23.2. Demostrar que, para $n \geq 2$, se verifica

$$\sum_{k=1}^{n^2-1}\left[\sqrt{k}\right]=\frac{n(n-1)(4n+1)}{6}.$$

Solución de Bruno Salgueiro Fanego (Viveiro, España):

Agrupando os sumandos que teñen a mesma parte enteira, resulta:

$$\sum_{k=1}^{n^2-1}\left[\sqrt{k}\right]=\left(\left[\sqrt{1}\right]+\left[\sqrt{2}\right]+\left[\sqrt{3}\right]\right)+\left(\left[\sqrt{4}\right]+\left[\sqrt{5}\right]+\left[\sqrt{6}\right]+\left[\sqrt{7}\right]+\left[\sqrt{8}\right]\right)$$
$$+\left(\left[\sqrt{9}\right]+\left[\sqrt{10}\right]+\left[\sqrt{11}\right]+\left[\sqrt{12}\right]+\left[\sqrt{13}\right]+\left[\sqrt{14}\right]+\left[\sqrt{15}\right]\right)$$
$$+\left(\left[\sqrt{4^2}\right]+\left[\sqrt{17}\right]+\left[\sqrt{18}\right]+\left[\sqrt{19}\right]+\left[\sqrt{20}\right]+\left[\sqrt{21}\right]+\left[\sqrt{22}\right]+\left[\sqrt{23}\right]+\left[\sqrt{5^2-1}\right]\right)+\cdots$$
$$+\left(\left[\sqrt{(m-1)^2}\right]+\left[\sqrt{(m-1)^2+1}\right]+\left[\sqrt{(m-1)^2+2}\right]+\cdots+\left[\sqrt{(m-1)^2+2m-4}\right]+\left[\sqrt{(m-1)^2+2m-3}\right]+\left[\sqrt{m^2-1}\right]\right)+\cdots$$
$$+\left(\left[\sqrt{(n-1)^2}\right]+\left[\sqrt{(n-1)^2+1}\right]+\left[\sqrt{(n-1)^2+2}\right]+\left[\sqrt{n^2-2n+4}\right]+\left[\sqrt{n^2-2n+5}\right]+\cdots+\left[\sqrt{n^2-2n+(2n-2)}\right]+\left[\sqrt{n^2-1}\right]\right)$$
$$=(1+1+1)+(2+2+2+2+2)+(3+3+3+3+3+3+3)+(4+4+4+4+4+4+4+4+4)$$
$$+\cdots+((m-1)+(m-1)+\cdots+(m-1))+\cdots+((n-1)+(n-1)+\cdots+(n-1))$$
$$=3\cdot 1+5\cdot 2+7\cdot 3+9\cdot 4+\cdots+((m-1)+(m-1)+\cdots+(m-1))+\cdots+((n-1)+(n-1)+\cdots+(n-1))$$
$$=(2\cdot 1+1)\cdot 1+(2\cdot 2+1)\cdot 2+(2\cdot 3+1)\cdot 3+(2\cdot 4+1)\cdot 4+\cdots+(2\cdot(m-1)+1)\cdot(m-1)+\cdots+(2\cdot(n-1)+1)\cdot(n-1)$$
$$=\sum_{m=1}^{n-1}(2\cdot m+1)\cdot m=2\sum_{m=1}^{n-1}m^2+\sum_{m=1}^{n-1}m=2\,\frac{(n-1)((n-1)+1)(2(n-1)+1)}{6}+\frac{1+(n-1)}{2}\cdot(n-1)$$
$$=\frac{n\cdot(n-1)}{6}\cdot(2\cdot(2n-1)+3)=\frac{n(n-1)(4n+1)}{6}.$$

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