PROPOSED PROBLEM TO REVISTA ESCOLAR DE LA OLIMPIADA IBEROAMERICANA DE MATEMATICA

OVIDIU FURDUI

Let $l \geq 0$ be a natural number. Find:

$$\sum_{n=1}^{\infty}(-1)^n \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln(n+2l+1)-\gamma\right),$$

where $\gamma = \sum\limits_{k=1}^{n} - \ln n,$ is the *Euler-Mascheroni's constant.*

Solution. The sum equals:

$$\frac{\gamma}{2}-\ln\frac{2^{l+1}l!}{(2l+1)!!\sqrt{\pi}}.$$

We need the following two well-known results.

Lemma 0.1. *Abel's summation by parts formula. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers and let $A_n = \sum\limits_{k=1}^{n} a_k$. The following formula holds:*

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k\left(b_k - b_{k+1}\right).$$

Proof. The lemma can be proven by elementary calculations. □

Lemma 0.2. *Walli's formula. The following limit holds:*

$$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

Let $S_n = \sum\limits_{k=1}^{n}(-1)^k\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}-\ln(k+2l+1)-\gamma\right)$. An application of Lemma 0.1, with $a_k = (-1)^k$ and $b_k = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}-\ln(k+2l+1)-\gamma$, shows that

$$\begin{aligned} S_n = {} & \left((-1)+(-1)^2+\cdots+(-1)^n\right)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}-\ln(n+2l+2)-\gamma\right)+ \\ & +\sum_{k=1}^{n}\left((-1)+(-1)^2+\cdots+(-1)^k\right)\left(\ln\frac{k+2l+2}{k+2l+1}-\frac{1}{k+1}\right). \end{aligned} \tag{0.1}$$

On the other hand, we have that

$$\lim_{n\to\infty}\left((-1)+(-1)^2+\cdots+(-1)^n\right)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}-\ln(n+2l+2)-\gamma\right)=0,$$

since $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}-\ln(n+2l+2)-\gamma\to 0$ as $n\to\infty$, and $A_n=(-1)+(-1)^2+\cdots+(-1)^n=0$, for n even and $A_n=-1$ for n odd.

Thus, letting n converge to ∞ in (0.1), we get that

$$S=\lim S_n=\sum_{k=1,k=odd}^{\infty}(-1)\left(\ln\frac{k+2l+2}{k+2l+1}-\frac{1}{k+1}\right)=-\sum_{p=0}^{\infty}\left(\ln\frac{2p+2l+3}{2p+2l+2}-\frac{1}{2p+2}\right).$$

Let T_n be the n^{th} partial sum of the preceding series, i.e.,

$$\begin{aligned}T_n&=\sum_{p=0}^{n-1}\left(\ln\frac{2p+2l+3}{2p+2l+2}-\frac{1}{2p+2}\right)\\&=\ln\frac{(2l+3)(2l+5)\cdots(2l+2n+1)}{(2l+2)\cdots(2n+2l)}-\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)-\ln\sqrt{n}.\end{aligned}$$

It follows that,

$$T_n=\ln\left(\frac{(2l)!!}{(2l+1)!!}\frac{(2n+2l-1)!!\sqrt{2n+2l+1}}{(2n+2l)!!}\sqrt{\frac{2n+2l+1}{n}}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right).$$

An application of Lemma 0.2 shows that

$$\lim_{n\to\infty}T_n=\ln\frac{(2l)!!}{(2l+1)!!}\sqrt{\frac{2}{\pi}}\sqrt{2}-\frac{\gamma}{2}=\ln\frac{2^{l+1}l!}{(2l+1)!!\sqrt{\pi}}-\frac{\gamma}{2}.$$

The desired result follows.

Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008

E-mail address: o0furdui@wmich.edu, ofurdui@yahoo.com

Revista Escolar de la Olimpíada Iberoamericana de Matemática

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