

PROPOSED SOLUTION TO PROBLEMA 141

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Sea $\binom{n}{i}$ un numero combinatorio, $n, i \in N$ $n \geq i$. Demostrar entonces que

$$\binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}$$

y $e^{\frac{1}{2}n^2}$ son dos infinitos asintoticamente equivalentes, es decir,

$$\lim_{n \rightarrow \infty} \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}} = 1.$$

Solution. In what follows we are going to prove that

$$\lim_{n \rightarrow \infty} \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}} = \infty.$$

We need the following Lemma.

Lemma 0.1. *Glaisher-Kinkelin constant. The following limit holds:*

$$A = \lim_{n \rightarrow \infty} \frac{1^1 2^2 \cdots n^n}{n^{n^2/2+n/2+1/12} e^{-n^2/4}},$$

where $A = 1.282712$ is the Glaisher-Kinkelin constant.

Let $x_n = \frac{e^{\frac{1}{2}n^2}}{\binom{n}{1} \binom{n}{2} \cdots \binom{n}{n}}$. We have

$$\ln x_n = \frac{n^2}{2} - \sum_{k=1}^n \ln \frac{n!}{k!(n-k)!} = \frac{n^2}{2} - (n+1) \ln n! + 2 \sum_{k=1}^n \ln k!.$$

On the other hand, we note that

$$\begin{aligned} \sum_{k=1}^n \ln k! &= n \ln 1 + (n-1) \ln 2 + \cdots + 2 \ln(n-1) + \ln n = \sum_{k=1}^n (n+1-k) \ln k \\ &= (n+1) \ln n! - \sum_{k=1}^n k \ln k. \end{aligned}$$

Thus,

$$\begin{aligned} \ln x_n &= \frac{n^2}{2} + (n+1) \ln n! - 2 \sum_{k=1}^n k \ln k \\ &= \frac{n^2}{2} + (n+1) \ln n! - 2 \left(\sum_{k=1}^n k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \right) \\ &\quad - \left(n^2 + n + \frac{1}{6} \right) \ln n + \frac{n^2}{2} = n^2 + (n+1) \ln n! - \left(n^2 + n + \frac{1}{6} \right) \ln n - 2A_n, \end{aligned}$$

where

$$A_n = \sum_{k=1}^n k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4}.$$

Using *Stirling's formula* $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, we get that

$$n^2 + (n+1) \ln n! - \left(n^2 + n + \frac{1}{6} \right) \ln n \approx (n+1) \ln \sqrt{2\pi n} - n - \frac{\ln n}{6}.$$

It follows, based on Lemma 0.1, that

$$\ln x_n \approx (n+1) \ln \sqrt{2\pi n} - n - \frac{\ln n}{6} - 2A_n \rightarrow \infty - 2 \ln A = \infty,$$

and hence $x_n \rightarrow \infty$.

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