Proposed solution of problem 155, vol.31 2008

Dear Editor of "Rivista Escolar de la Olimpiada Iberoamericana de Matemática", I would like to submit the following solution of problem 155

Sean $a; b; c$ números reales no negativos. Calcular el límite

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k+a}{\sqrt{n^2+nk+b}\,\sqrt{n^2+nk+c}}$$

Answer: $1-\ln 2$

Proof We shall employ $\sum\limits_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+o(1)$ and γ is the well known Euler–Mascheroni constant

$$0<\sum_{k=1}^{n}\frac{a}{\sqrt{n^2+nk+b}\,\sqrt{n^2+nk+c}}=\sum_{r=n+1}^{2n}\frac{a}{\sqrt{nr+b}\,\sqrt{nr+c}}=$$
$$=\sum_{r=n+1}^{2n}\frac{a}{nr}\frac{1}{\sqrt{1+\frac{b}{nr}}\,\sqrt{1+\frac{c}{nr}}}\le\sum_{r=n+1}^{2n}\frac{a}{nr}=\frac{a}{n}(\ln(2n)-\ln(n+1)+o(1))$$

and the limit $n\to\infty$ is zero.

$$\sum_{k=1}^{n}\frac{k}{\sqrt{n^2+nk+b}\,\sqrt{n^2+nk+c}}=\sum_{r=n+1}^{2n}\frac{r-n}{\sqrt{nr+b}\,\sqrt{nr+c}}\ ;$$

$$\sum_{r=n+1}^{2n}\frac{n}{\sqrt{nr+b}\,\sqrt{nr+c}}=\sum_{r=n+1}^{2n}\frac{1}{r}\frac{1}{\sqrt{1+\frac{b}{nr}}\,\sqrt{1+\frac{c}{nr}}}=$$
$$=\sum_{r=n+1}^{2n}\frac{1}{r}\big((1+O(\frac{b}{nr}))(1+O(\frac{c}{nr}))\big)=\sum_{r=n+1}^{2n}\frac{1}{r}\big(1+O(\frac{1}{nr})\big)=$$
$$=(\ln(2n)-\ln(n+1)+o(1))+O(n^{-1})$$

whose limit is $\ln 2$.

$$\sum_{r=n+1}^{2n}\frac{r}{\sqrt{nr+b}\,\sqrt{nr+c}}=\sum_{r=n+1}^{2n}\frac{1}{n}\frac{1}{\sqrt{1+\frac{b}{nr}}\,\sqrt{1+\frac{c}{nr}}}=$$
$$=\sum_{r=n+1}^{2n}\frac{1}{n}\big((1+O(\frac{b}{nr}))\cdot(1+O(\frac{c}{nr}))\big)=\sum_{r=n+1}^{2n}\frac{1}{n}\big(1+O(\frac{1}{nr})\big)=$$
$$=1+O(n^{-1})$$

whence the result announced. The proof is completed.

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If you will publish or simply acknowledge my solution, please, mention "Dipartimento di matematica" or "math. dept." or anything you like with this information. In the Rome–area there are two "Paolo Perfetti" and although one (me) is a mathematician while the second is a physicist, often it is unclear which one is cited.

Roma(Italy) 04/08/08

Best regards
Paolo Perfetti

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