PROBLEMA 180:

Sea $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Demostrar que

$$\frac{H_1}{2} - \frac{H_2}{3} + \frac{H_3}{4} - \cdots + (-1)^{n+1}\frac{H_n}{n+1} + \cdots = \frac{(\ln 2)^2}{2}.$$

SOLUCION:

Primero probaremos si la serie alternante converge

i) $\frac{H_n}{n+1} \geq \frac{H_{n+1}}{n+2} \iff H_n(n+2) \geq H_{n+1}(n+1) = \left(H_n + \frac{1}{n+1}\right)(n+1) = H_n(n+1) + 1 \iff H_n(n+2-n-1) \geq 1 \iff H_n \geq 1$ se cumple para cualquir n natural

ii) Como $\sum_{i=1}^{n} \frac{1}{i} = 1 + \sum_{i=2}^{n} \frac{1}{i} < 1 + \int_1^n \frac{1}{x} dx = 1 + \ln n$ entonces

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}\frac{1}{i}}{n+1} \leq \lim_{n\to\infty} \frac{1+\ln n}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 \implies \frac{H_n}{n+1} \longrightarrow 0 \text{ cuando } n \to \infty$$

Por el criterio de la serie alternante, esta serie converge.

Ahora proseguimos a calcular su valor.

Sea $S = \frac{H_1}{2} - \frac{H_2}{3} + \frac{H_3}{4} - \cdots + (-1)^{n+1}\frac{H_n}{n+1} + \cdots$ al agruparla de la siguiente forma

$S = \frac{1}{2}(1) - \frac{1}{3}\left(1 + \frac{1}{2}\right) + \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \cdots + \frac{(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) + \cdots = \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + (-1)^n \frac{1}{n} + \cdots\right)(1) + \left(-\frac{1}{3} + \frac{1}{4} - \cdots + (-1)^n\frac{1}{n} + \cdots\right)\left(\frac{1}{2}\right) + \cdots + \left((-1)^{n+1}\frac{1}{n+1} + (-1)^{n+2}\frac{1}{n+2} + \cdots\right)\left(\frac{1}{n}\right) + \cdots = (1)\sum_{j=2}^{\infty}(-1)^j\frac{1}{j} + \left(\frac{1}{2}\right)\sum_{j=3}^{\infty}(-1)^j\frac{1}{j} + \cdots + \left(\frac{1}{n}\right)\sum_{j=n+1}^{\infty}(-1)^j\frac{1}{j} + \cdots = \sum_{i=1}^{\infty}\frac{1}{i}\sum_{j=i+1}^{\infty}(-1)^j\frac{1}{j} = \sum_{i=1}^{\infty}\frac{1}{i}\sum_{j=i+1}^{\infty}(-1)^j\int_0^1 x^{j-1}\,dx = \sum_{i=1}^{\infty}\frac{1}{i}\int_0^1\sum_{j=i+1}^{\infty}(-1)^j x^{j-1}dx$ pero

$$\sum_{j=i+1}^{\infty}(-1)^j x^{j-1} = -\sum_{j=i+1}^{\infty}(-1)^{j-1}x^{j-1} = -\sum_{j=i+1}^{\infty}(-x)^{j-1} = -(-x)^i\sum_{k=0}^{\infty}(-x)^k$$
$$= -(-x)^i\left(\frac{1}{1+x}\right)$$

Sustituyendo

$$S = \sum_{i=1}^{\infty}\frac{1}{i}\int_0^1 -(-x)^i\left(\frac{1}{1+x}\right)dx = -\int_0^1\sum_{i=1}^{\infty}\frac{(-x)^i}{i}\left(\frac{1}{1+x}\right)dx = -\int_0^1\frac{1}{1+x}\sum_{i=1}^{\infty}\frac{(-x)^i}{i}\,dx$$

Como $\sum_{i=1}^{\infty}\frac{(-x)^i}{i} = \ln(1+x)$ sustituyendo se obtiene

$$S = \int_0^1 \frac{1}{1+x}\ln(1+x)\,dx$$

Utilizando integración por partes con,

$u = \ln(1+x) \quad y \quad dv = \frac{1}{1+x}dx \,, du = \frac{1}{1+x}dx \quad y \quad v = \ln(1+x)$

Se tiene $\int \frac{1}{1+x}\ln(1+x)\,dx = [\ln(1+x)]^2 - \int \frac{1}{1+x}\ln(1+x)\,dx \Longrightarrow \int \frac{1}{1+x}\ln(1+x)\,dx = \frac{[\ln(1+x)]^2}{2}$

Evaluando

$$S = \frac{[\ln(2)]^2}{2}$$